New ways for analysis of interferograms in phase-shifting interferometry

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ABSTRACT

The basic concepts in the application of new methods to interferogram analysis are introduced. Three ideas are underlined: first, way of obtaining decoding of the equations for the analysis of interferograms at arbitrary phase shifts, second, definition of actual values of phase shifts immediately from values of brightness of interferograms and third, method of a volume expansion at measurement of a phase difference of light waves.

Keywords: Interferometry, Interferogram analysis, Phase shift

1. INTRODUCTION

Almost all major problems of interferometry are connected with fringe analysing procedures: fringe peak detection, fringe order determination, the fractional fringe order, etc. As an alternative to classical methods of interferometry there appeared phase shifting interferometry. The latter is based on measurement of intensity at given points under the change of phase shift in an interferometer aim. One came determine the phase difference of two interfering waves from the correlation:

$$I = A + V\cos\phi \tag{1}$$

where I is the value of the intensity at the point being analysed, A initial intensity at this point, V is the visibility of interference fringes, ϕ is the value of phase. If one has A, upon measuring I and V, one can obtain the value of ϕ from Eq.(1). Evidently, this value can be determined within the limit of one period only. However, this method of phase determination is practically unacceptable due to variability of V and I. Then Eq. (1) is obtained:

$$I_i = A + V\cos(\phi + \delta_i)\,,\ i \in 0,\ldots,N\,. \tag{2}$$

This method is well known as a phase-shifting interferometry (PSI).[1] In a case, when $\delta_i = (i+1)\delta_0$, i.e. $\delta_1 = 2\delta_0$, $\delta_2 = 3\delta_0$ etc. the solution of a set of equations (2) can be found by a Fourier-series expansion[2]

$$\phi = \arctan\frac{\sum_i I_i \cdot \sin(i\delta_i)}{\sum_i I_i \cdot \cos(i\delta_i)} \tag{3}$$

Traditionally, the phase shifts δ_i select identical so it reduces in simpler sort of a solution of a set of equations (2). However, as a matter of fact to place shifts precisely it is impossible and then there are errors at a solution of the system (3). In this case it is necessary to solve (2) immediately that at major number of shifts causes definite difficulties.

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Seventh International Symposium on Laser Metrology Applied to Science, Industry, and Everyday Life,
Yuri V. Chugui, Sergei N. Bagayev, Albert Weckenmann, P. Herbert Osanna, Editors,
Proceedings of SPIE Vol. 4900 (2002) © 2002 SPIE · 0277-786X/02/$15.00

2. GENERALIZED ALGORITHM OF DECODING OF INTERFEROGRAMS WITH STEP SHIFT

In this section the approach to build-up of resolving algorithms is considered at phase shifts having an arbitrary relation. Let's expression (2) given in the vector's form:

$$\vec{I} = A\vec{R} + (V\cos\phi)\vec{C} + (V\sin\phi)\vec{S} \tag{4}$$

Where $\vec{R} = (1,\ldots,1)^T$, $\vec{C} = (\cos\delta_0,\ldots,\cos\delta_N)^T$, $\vec{S} = (\sin\delta_0,\ldots,\sin\delta_N)^T$, the dimension of vectors is defined by number of phase shifts δ_N. Is foolproof to show, that the solution of a set of equations (4) can be received, selecting quadrature component, then within a constant factor V:

$$\sin\phi = \frac{\vec{I}\cdot\vec{C}^{\perp}}{\vec{S}\cdot\vec{C}^{\perp}} \tag{5a}$$

both

$$\cos\phi = \frac{\vec{I}\cdot\vec{S}^{\perp}}{\vec{C}\cdot\vec{S}^{\perp}}, \tag{5b}$$

where the operator $(a\cdot b)$ - means a dot product of vectors, and $\vec{S}^{\perp}$ and $\vec{C}^{\perp}$ - vector, orthogonal vectors $\vec{S}$, $\vec{R}$ and $\vec{C}$, $\vec{R}$ accordingly. Allowing known property of a dot product $(a\cdot b)=(a\cdot b)$ the algorithm of decryption (4) in the vector's form start. Following sort

$$\phi = \arctan\frac{\vec{I}\cdot\vec{C}^{\perp}}{\vec{I}\cdot\vec{S}^{\perp}} \tag{6a}$$

And, taking into consideration property of a vector product $(a\times b) = -(b\times a)$, the calculation of expression (6) can be simplified as

$$\phi = \arctan\frac{\vec{I}^{\perp}\cdot\vec{C}}{\vec{I}^{\perp}\cdot\vec{S}} \tag{6b}$$

Since in this case it is required follow-up to compute only one vector $\vec{I}^{\perp}$. For this purpose it is convenient to use a matrix operator $\vec{I}^{\perp} = M\cdot\vec{I}$. The transformation matrix, should obey which one to the following requirements: $(M\cdot\vec{I})\cdot\vec{I} = 0$ and $M\cdot\vec{R} = 0$. In a case $i = 3$ matrix M - skew-symmetric matrix of sort:

$$M = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{7}$$

For example, at $i = 3$ from Eq. (6) the following algorithm of decryption is received

$$\phi = \arctan\frac{(M\cdot\vec{I})\cdot\vec{C}}{(M\cdot\vec{I})\cdot\vec{S}} = \arctan\frac{(I_1 - I_2)\cdot c_0 + (I_2 - I_0)\cdot c_1 + (I_0 - I_1)\cdot c_2}{(I_1 - I_2)\cdot s_0 + (I_2 - I_0)\cdot s_1 + (I_0 - I_1)\cdot s_2}. \tag{8}$$

In case of an odd amount of phase shifts $i > 3$ the matrix is m gained by symmetric prolongation of a matrix (7)

$$M = \left[\begin{array}{ccc:ccc} 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & 0 & 1 & -1 & 1 \\ \hdashline -1 & 1 & -1 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 1 & -1 & 0 \end{array}\right]. \tag{9}$$

For example, at $i = 5$ the algorithm of decryption accepts following sort

$$\phi = \arctan \frac{(I_1 - I_2 + I_3 - I_4)\cdot c_0 + (I_2 - I_0 + I_4 - I_3)\cdot c_1 +}{(I_1 - I_2 + I_3 - I_4)\cdot s_0 + (I_2 - I_0 + I_4 - I_3)\cdot s_1 +}\cdots$$

$$\cdots \frac{+(I_0 - I_1 + I_3 - I_4)\cdot c_2 + (I_1 - I_0 + I_4 - I_2)\cdot c_3 + (I_0 - I_1 + I_2 - I_3)\cdot c_4}{+(I_0 - I_1 + I_3 - I_4)\cdot s_2 + (I_1 - I_0 + I_4 - I_2)\cdot s_3 + (I_0 - I_1 + I_2 - I_3)\cdot s_4}. \tag{10}$$

At an even amount of phase shifts the matrix M can be presented as

$$M = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix},\ B = \left[\begin{array}{cc:cc} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ \hdashline -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{array}\right]. \tag{11}$$

Then, for example, at $i = 4$ the following algorithm of decryption is received

$$\phi = \arctan \frac{(I_2 - I_3)\cdot(c_1 - c_0) + (I_1 - I_0)\cdot(c_2 - c_3)}{(I_2 - I_3)\cdot(s_1 - s_0) + (I_1 - I_0)\cdot(s_2 - s_3)}. \tag{12}$$

On the other hand, at number of phase shifts more than three, except for independent solutions of the equations of decryption (5) or (6) are possible to receive their linear combinations from three-component sub-vectors $I^{(\bullet)}$ and appropriate by them sub-vectors $C^{(\bullet)}$ and $S^{(\bullet)}$ accordingly

$$\phi = \arctan \frac{\sum_m I^{(m)} C^{\perp(m)}}{\sum_m I^{(m)} S^{\perp(m)}} = \arctan \frac{\sum_m I^{\perp(m)} C^{(m)}}{\sum_m I^{\perp(m)} S^{(m)}}, \tag{13}$$

where: $m = M_N^3 = \dfrac{N!}{3!(N-3)!}$ - number of optional versions of «triples» from N of points. For example, at $N = 4$ the following combinations sub-vectors $I^{(m)}$ are possible

$$I \to \begin{bmatrix} I_0 \\ I_1 \\ I_2 \\ I_3 \end{bmatrix} : I^{(1)} \to \begin{bmatrix} I_0 \\ I_1 \\ I_2 \end{bmatrix}, I^{(2)} \to \begin{bmatrix} I_0 \\ I_1 \\ I_3 \end{bmatrix}, I^{(3)} \to \begin{bmatrix} I_0 \\ I_2 \\ I_3 \end{bmatrix}, I^{(4)} \to \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}.$$

For example, at a combination sub-vectors $I^{(2)}$ and $I^{(3)}$ from Eq. (13) the following algorithm of decryption is received

$$\phi = \arctan\frac{(I_2+I_1-2I_3)\cdot c_0+(I_3-I_0)\cdot(c_1+c_2)+(2I_0-I_1-I_2)\cdot c_3}{(I_2+I_1-2I_3)\cdot s_0+(I_3-I_0)\cdot(s_1+s_2)+(2I_0-I_1-I_2)\cdot s_3}. \quad (14)$$

Let's mark, that if the phase shifts have the aliquot value, the surveyed algorithms of decoding pass immediately in Eq. (2). Thus, the generalized algorithm permitting to gain a solution of a problem of decoding at arbitrary phase shifts is obtained.

3. ANALYSIS OF ARBITRARY PHASE-SHIFT INTERFEROGRAMS

In the section we consider a new decoding method based on determination of phase increments from available interferograms. In this case, one may set arbitrary shifts. Hence, preliminary correction of phase-shifting devices becomes unnecessary. It is seen from (2) that the unknown angle of shift 5, is added to the three unknowns A, V and ϕ. In this case, there are 3 + m -1 unknowns $A, V, \delta_0, \delta_1 \cdots \delta_m$; ($\delta_0 = 0$) at each point. If n is the number of taken points, then the total number of unknowns is $3n+m-1$ for $n \times m$ equations. The solution can be found if the total number of equations is greater than or equal to the number of unknowns, i.e.,

$$n \times m \geq 3n-1 \qquad \text{or} \qquad n \geq 1+\tfrac{2}{m-3} \quad (15)$$

This can be attained assuming that the angles of shift are equal at the neighboring points. This assumption is fulfilled in most cases on the basis of physical experimental conditions.

There are two possible cases for which the number of unknowns is equal to the number of equations: at two points for five shifts and at three points for four shifts. In all of the rest variants the number of unknowns is either less or greater than the number of equations. In the present paper we consider the analytical solution when recording five interferograms with the phase shifts δ_i, i = 0,…,4 under assumption of equal phase shifts at two neighboring points. In this case we obtain ten transcendent equations with ten unknowns. Let n = 2, m= 5, then from (1) and (2) follows

$$\begin{cases} I_{1i} = A_1 + V_1\cos(\phi+\delta_i) \\ I_{2i} = A_2 + V_2\cos(\phi+\varepsilon+\delta_i) \end{cases}, \; i = 0,\ldots,4 \quad (16)$$

(here ε is the phase difference between two points on the interferogram).

We will seek for the solution in the complex plane with the axes I_1 and I_2 on which the values of intensities of the first and second points, corresponding to different phase shifts, are plotted. As the angles of shift vary from 0 up to 2π, the point on the complex plane describes any trajectory. It can be easily shown that the trajectory is a central curve of the second order (an arbitrarily oriented ellipse) with the center at the point (A_1, A_2). Using the well-known formulas from analytical geometry[3] and in view of the accepted notation, we will write the equation of the trajectory in the form

$$a_{11}x^2+2a_{12}xy+a_{22}y^2+2a_{13}x+2a_{14}y+a_{33}=0 \quad (17)$$

The coefficients a_{ij} from (17) can be obtained by calculating the determinant of the minor at the corresponding coefficient:

$$\begin{vmatrix} a_{11} & a_{12} & a_{22} & a_{13} & a_{14} & a_{33} \\ I_{11}^2 & I_{11}I_{21} & I_{21}^2 & I_{11} & I_{21} & 1 \\ I_{12}^2 & I_{12}I_{22} & I_{22}^2 & I_{12} & I_{22} & 1 \\ I_{13}^2 & I_{13}I_{23} & I_{23}^2 & I_{13} & I_{23} & 1 \\ I_{14}^2 & I_{14}I_{24} & I_{24}^2 & I_{14} & I_{24} & 1 \\ I_{15}^2 & I_{15}I_{25} & I_{25}^2 & I_{15} & I_{25} & 1 \end{vmatrix} \quad (18)$$

For example, the coefficient a_{11} is equal to the determinant obtained from (18) by deleting the first row and the first column; a_{12} is equal to the determinant taken with the sign «-» by deleting the first row and the second column, etc. The value of average brightness $A = (A_1, A_2)$ corresponding to the coordinates of the center of the curve (17) is found from the set of linear equations in the following manner:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} -a_{13} \\ -a_{14} \end{bmatrix} \quad (19)$$

The values of contrast of the interferograms V_1, V_2 - can be obtained from the following relationships:

$$d_1^2 + d_2^2 = V_1^2 + V_2^2 \quad (20)$$

$$\frac{a_{22}}{a_{11}} = \frac{V_2^2}{V_1^2} \quad (21)$$

where d_1 and d_2 - are the principal axes of the ellipse (17). The parameters of the principal axes are defined by the invariant $\tilde{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{12} \end{vmatrix}$ and the eigenvalues - λ_1, λ_2 being the solution of equation obtained from the characteristic matrix composed of the coefficients (17):

$$\begin{vmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{12} & a_{12} - \lambda_2 \end{vmatrix} = 0 \quad (22)$$

Joint solution of (20), (21) and (22) yields:

$$V_1 = \sqrt{\frac{a_{11}}{a_{11} + a_{22}} \cdot \frac{-\tilde{A}(\lambda_1 + \lambda_2)}{(\lambda_1 \lambda_2)^2}} \quad (23)$$

$$V_2 = \sqrt{\frac{a_{22}}{a_{11} + a_{22}} \cdot \frac{-\tilde{A}(\lambda_1 + \lambda_2)}{(\lambda_1 \lambda_2)^2}} \quad (24)$$

By substituting the retrieved unknowns in (3) it is possible to receive the system of simple equations, which one can be easily solved of the rather stayed unknowns[4]. However this path reduces in necessity of prior knowledge of a direction of a phase shift. Let's show one more possible way of a solution, which one does not require fulfillment of the given condition. Now, we shall map points of the ellipse on a single circle. It is possible to make if to transfer center of the ellipse in a beginning of coordinates, to turn a principal axis of the ellipse so that a point $\widetilde{I}_0 = (I1_0, I2_0)$ appropriate $\delta_0 = 0$ was mated with an abscissa axis and to carry out tension (or compression) of coordinate appropriate major axis of the ellipse on coefficient proportional $\frac{V_1}{V_2}$. The indicated conversions are fulfilled as follows:

$$\begin{bmatrix} \widetilde{I}_{1i} \\ \widetilde{I}_{2i} \end{bmatrix} = \frac{V_1}{V_2} \cdot \begin{bmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{bmatrix} \cdot \begin{bmatrix} I_{1i} - A_1 \\ I_{2i} - A_2 \end{bmatrix} \tag{25}$$

where $\alpha = -\frac{1}{2}\arctan\left(\frac{2a_{12}}{a_{22} - a_{11}}\right)$.

Then the corners of phase shifts will correspond to corners between vectors $\widetilde{I}_i = (\widetilde{I}_{1i}, \widetilde{I}_{2i})$. Further, retrieved the corners of phase shift are substituted in the equation of decoding (13) and the phase difference of wave front sets is determined.

The described algorithm allows decoding of interferograms with arbitrary phase shifts. Thus, we have proposed a new method for interferogram analysis and obtained an algorithm for eliminating the main cause of the error in measurements by phase-shifting interferometry, that is, the error of wrong phase shift setting.

4. INTEGER INTERFEROMETRY

The interferometry is aimed at obtaining the total difference of the wave front phases from a two-dimensional interference pattern or from a periodic variation of brightness at the given points. The decoding techniques based on controlled phase shift have been applied recently. The primary reasons for the acceptance of the phase-shifting interferometry are the ease of data analysis and the extreme precision to which measurements can be made. Other advantages of this technique are the ability to determine the phase difference at any point of the field being analyzed. The phase difference, however, is determined to the precision of the wavelength as with other interference techniques. The total phase will be correctly reconstructed only if the phase difference between adjacent measurement points is a half wave or less. This condition dramatically limits the type of the wave fronts being measured[5]. To overcome this they use null lenses, computer-generated holograms, and the increased number of pixels in detector arrays of the interferogram input devices, the longer wavelengths of the light sources, the two-wavelength techniques. However these techniques either make the system much more complex or lead to its decreased precision.

The purpose of this section is to describe a new method of interferogram analysis based on the use of the natural number properties that allows determination of the total phase at effective wavelength variation. The method is based on divisibility treated by the number theory. If the integers leave the same remainder r on division by m, they are called congruent modulo m m. To express this fact one writes:

$$a = b \pmod{m} \tag{26}$$

The numbers congruent modulo m make the number class modulo m, all the class numbers being obtained from the expression $mq + r$, where q takes on the values of all the integers.

Any class number is called the residue modulo m. The residues giving all possible values of r make the whole residue system modulo m.

If in comparison (26) one of the numbers, say, a, is unknown, then by calling it x one can write

$$x \equiv b \pmod{m} \tag{27}$$

The expression (27) is a comparison of the 1st order with one unknown. To solve this comparison means to find all the satisfactory x values» If an $x = x^*$ satisfies the comparison (27), it will satisfy all the numbers compared with x^* modulo m. Hence the whole class of numbers determined by the comparison

$$x \equiv x^* \pmod{m} \tag{28}$$

will be the solution of comparison (27). Consider the solution of the simultaneous comparisons of the 1st order

$$\begin{aligned} x &= b_1 \pmod{m_1} \\ x &= b_2 \pmod{m_2} \\ &\cdots \\ x &= b_k \pmod{m_k} \end{aligned} \tag{29}$$

modulo being not equal to each other and being prime numbers. The solution of the simultaneous comparison (29) is the number class (28), which satisfies all the comparisons simultaneously. It's necessary to find x^* and m. In the number theory a theorem is proved according to which

$$x^* = M_1 M_1^* b_1 + M_2 M_2^* b_2 + \ldots + M_k M_k^* b_k, \tag{30}$$

$$m = m_1 m_2 \cdots m_k, \tag{31}$$

where M_k and M_k^* are the numbers found in the conditions

$$M_s m_s = m_1 m_2 \cdots m_k \quad , \tag{32}$$

$$M_s m_s = 1 \pmod{m_1}, \tag{33}$$

where s takes on the values from 1 to k in succession.

The properties of simultaneous comparison solution (29) are the basis of the interferogram technique proposed. By approaching the result obtained in the number theory to the task being solved in the interferometry it can be interpreted as follows. If linear periodic functions are given by their entire values and their periods are mutually prime numbers, then a linear periodic function with the period equal to the product of the periods of these functions can also be put them in a single-valued correspondence.

Let's determine the phase difference by phase-shifting interferometry with one effective wavelength. The phase difference values within a period can be put in correspondence to the integers b with the same valid digits and with the number of digits depending on the measurement accuracy. The period determined by the effective wavelength can be put in correspondence to the integer m with the same number of digits.

A series of measurements with other effective wave length values such as their periods can be put in correspondence to mutually prime numbers being carried on, x^* (30) corresponding to the total phase can be unambiguously be found within the dynamic range determined in (31) and (32), (33).

The effective wavelength can vary due to variation in the wavelength, the refractive index of medium and that of angle at wave front interference.

Let's consider a condition when measurements are carried on with two wavelength values. They are 529 nm and 633 nm. We put the integers 529 and 633 to their correspondence. They are mutually prime numbers. Then the number of 529*633=334857, which makes 529 times 633 periods, determines the measurement range. If we consider more decimals or carry on measurements with three or more period values, the gain can be higher.

The main advantage of the method proposed is a substantial increase of the dynamic range thereby high accuracy of measurements characteristic of the phase-shifting interference techniques and independence of the results from measurements at adjacent points are maintained. The technique doesn't require introduction of additional elements into the interferometer circuit and allows complete automation of the interferogram decoding.

ACKNOWLEDGEMENTS

This work was made under the support of INTAS grant № 00135.

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