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Automatic Processing of Fringe Patterns in Integer Interferometers

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ABSTRACT

The purpose of this paper is to describe a new method for determination of the total phase difference of light waves without interference fringe counting. One can build up an interferometer for measurement of displacements in which the displacements are determined only by the final values of intensity regardless of the velocity and any prior displacements that caused the present one. The same thing holds true for vibration, relief parameters and other values to be measured. Such an interferometer is devisable thanks to two underlying ideas: first, the application of controllable phase shift, and secondly, the use of properties of integer divisability.

PHASE SHIFTING INTERFEROMETRY

Almost all major problems of interferometry are connected with fringe analysing procedures: fringe peak detection, fringe order determination, ambiguity in fringe orders, the fractional fringe orders, etc.

As an alternative to classical methods of interferometry there appeared phase shifting interferometry. The latter is based on measurement of intensity at given points under the changes of phase shift in an interferometer arm.

One can determine the phase difference of two interfering waves from the correlation:

$$I = I_0(1 + V\cos\varphi) \tag{1}$$

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where I is the value of the intensity at the point being analyzed, I_0 is the initial intensity at this point, V is the visibility of interfringes, φ is the value of the phase.

If one has I_0, upon measuring I and V, one can obtain the value of φ from eqn (1). Evidently, this value can be determined within the limits of one period only.

However, this method of phase determination is practically unacceptable due to variability of I_0 and V. Then eqn (1) is obtained, for example, for three given values of phase ψ_i:

$$I_i = I_0(1 + V\cos(\varphi - \psi_i)) \qquad i = 1, 2, 3 \tag{2}$$

Three values of intensity are measured, I_1, I_2, I_3. Solving the system of three equations (2) for φ, one has[1]

$$\tan\varphi = \frac{(I_3 - I_2)\cos\psi_1 + (I_1 - I_3)\cos\psi_2 + (I_2 - I_1)\cos\psi_3}{(I_3 - I_2)\sin\psi_1 + (I_1 - I_3)\sin\psi_2 + (I_2 - I_1)\sin\psi_3} \tag{3}$$

For $\psi_1 = 0$, $\psi_2 = 2\pi/3$ and $\psi_3 = 4\pi/3$ one obtains

$$\operatorname{tg}\varphi = \frac{\sqrt{3}(I_3 - I_2)}{2I_1 - I_2 - I_3} \tag{4}$$

The method of phase calculation in eqns (2) and (3) is rather sensitive to phase shifting errors. Therefore other methods with self-correction were suggested, where real values of phase shifts are evaluated. One of them is based on four intensity measurements for the phase shifts: -3α, $-\alpha$, α, 3α. In this case one has[2]

$$\tan\varphi = \operatorname{tg}\alpha\,\frac{I_1 + I_2 - I_3 - I_4}{I_1 - I_2 - I_3 + I_4} \tag{5}$$

$$\tan\alpha = \sqrt{\frac{3I_2 - 3I_3 - I_1 + I_4}{I_1 + I_2 - I_3 - I_4}} \tag{6}$$

This method of phase calculation (eqns (5) and (6)) gives a large error for the phase determination, when $\varphi \sim k\pi$, where k is integer. Another method with five intensity measurement for the phase shifts -2α, $-\alpha$, 0, α, 2α does not have this disadvantage.[3] For α close to $\pi/2$ the phase is calculated from the correlation:

$$\tan\varphi = \frac{2(I_2 - I_4)}{2I_3 - I_1 - I_5} \tag{7}$$

To decrease the probable errors one can determine the phase φ, solving the system of n equations of the kind in eqn (2) by the least squares method. Then the solution is as follows:[4,5]

$$\tan \varphi = \frac{\sum_{i=1}^{n} I_i \sin \psi_i}{\sum_{i=1}^{n} I_i \cos \psi_i} \tag{8}$$

where $\psi_i = 2\pi(i-1)/n$, $i = 1, 2, \ldots, n$.

For $n = 4$ ($\psi_i = 0, \pi/2, \pi, 3\pi/2$) eqn (8) is very simple:[6]

$$\tan \varphi = \frac{I_2 - I_4}{I_1 - I_3} \tag{9}$$

It must be stressed that for phase determination one does not need to count down the interference fringe; measurement of the intensity at the point under study will suffice.

The appearance of phase shifting interferometry has been quite an event due to such recognized advantages as accuracy of measurements, independence of the results from measurements at adjacent points, complete automation of the decoding of the fringe patterns with any shape and degree of complexity. However, this method has a disadvantage of principal importance, the so-called indeterminance to a factor of 2π, or 2π ambiguities. Hybrid methods came into being which presented a kind of compromise with classical interferometry, as fringes were again used to eliminate indeterminance to a factor of 2π.

INTEGER INTERFEROMETRY

The method described in this paper makes it possible to determine the total phase difference with no interfringe counting by means of the application of phase shifting interferometry.

The method is based on the divisibility properties of integers treated by the number theory.[7,8] Here are the basic data indispensable for understanding the idea of the method.

If integers, a and b, leave the same remainder on division by m, they are called congruent modulo m. To express this fact one writes:

$$a \equiv b (\text{mod}\, m) \tag{10}$$

For example, 2 is congruent with 7 modulo 5, as, on division by 5, the numbers 2 and 7 leave the same remainder $r = 2$. All these numbers $2, 7, 12, \ldots, 5q + 2$, where q is any integer, make the number class modulo 5. Any class number is called the residue modulo 5. The

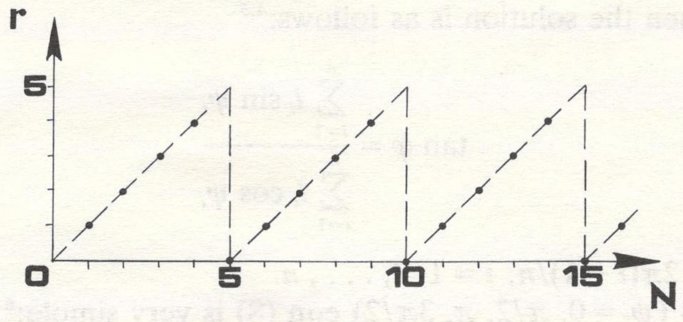


Fig. 1. Complete residue systems modulo 5.

residues giving all possible remainders $r = 0, 1, 2, \ldots, m-1$ make up a complete residue system modulo m. This can be presented graphically.

Figure 1 shows a graph for $m = 5$. On the N-axis there is a natural set of numbers. The complete residue systems are marked with points. They are reiterated with a period equal to the modulo m and are reflexed by an integer saw-shaped function.

If, in congruence (eqn (10)), one of the numbers is unknown, then, by calling it X, one can write:

$$X \equiv b(\operatorname{mod} m) \tag{11}$$

If some $X = X_0$ satisfies the congruence (eqn (11)), it will be satisfied by all

$$X \equiv X_0(\operatorname{mod} m) \tag{12}$$

This class of numbers will be the solution of the congruence. Of interest is the solution of the congruence system of the kind in eqn (11):

$$\begin{aligned} X &\equiv b_1(\operatorname{mod} m_1) \\ X &\equiv b_2(\operatorname{mod} m_2) \\ &\vdots \\ X &\equiv b_k(\operatorname{mod} m_k) \end{aligned} \tag{13}$$

the moduli being not equal to each other and are not relative primes.

It is necessary to find a solution, eqn (12), which would satisfy all the congruences in eqn (13) simultaneously.

In number theory a theorem is proved according to which

$$X_0 = M_1M'_1b_1 + M_2M'_2b_2 + \cdots + M_kM'_kb_k \tag{14}$$

$$m = m_1m_2 \cdots m_k \tag{15}$$

where M_k and M'_k are the numbers found in the conditions:

$$M_S m_S = m_1 m_2 \cdots m_k \tag{16}$$

$$M_S M'_S \equiv 1 (\mathrm{mod}\, m_S) \tag{17}$$

where S takes on values from 1 to k in succession.

Consider one simple example.

Solving the system of two congruences

$$\begin{aligned} X &\equiv b_1 (\mathrm{mod}\, 4) \\ X &\equiv b_2 (\mathrm{mod}\, 5) \end{aligned} \tag{18}$$

we obtain from eqns (14)–(17):

$$\begin{array}{ll} M_1 = 5 & M_2 = 4 \\ M'_1 = 1 & M'_2 = 4 \\ m = 20 & X_0 = 5b_1 + 16b_2 \end{array}$$

and the result is as follows:

$$X = 5b_1 + 16b_2 (\mathrm{mod}\, 20) \tag{19}$$

If one displays the congruence system, eqn (18), and its solution, eqn (19), graphically (Fig. 2), then each congruence of the system presents an integer saw-shaped function with the periods $m_1 = 4$ and $m_2 = 5$, respectively (like the graph in Fig. 1), and the solution is the same function but with the period $m = 20$.

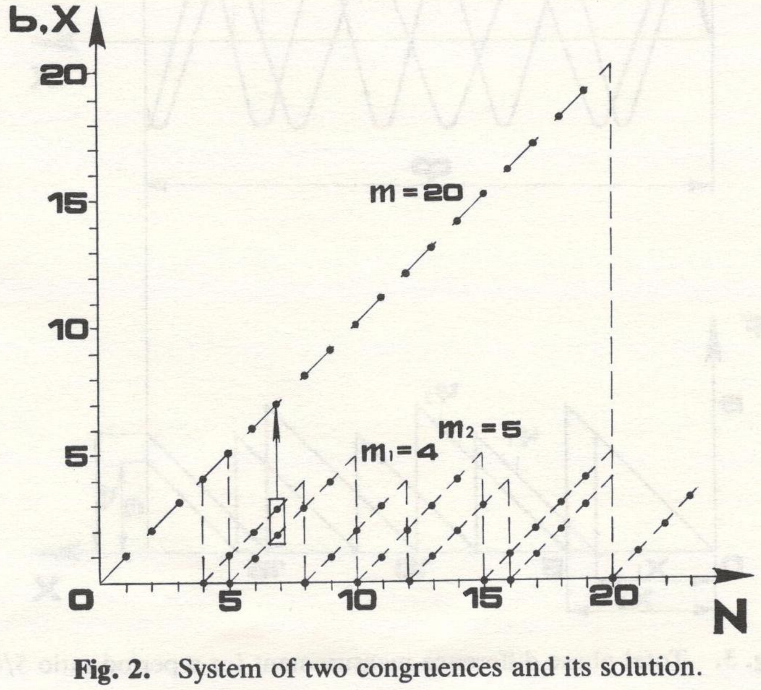


Fig. 2. System of two congruences and its solution.

For example, for $b_1 = 3$ and $b_2 = 2$ one has $X_0 = 47$ and $X = |47|_{20} = 7$, as shown in Fig. 2.

How is the above mentioned mathematics related to interferometry? The point is that the phase of the light wave is a linear function of the coordinate, and the integer functions discussed above are also linear functions. Mathematical methods of solution of integer congruences prove to be adequate to the task of the total phase difference determination in interferometry.[9–11] By approaching the result obtained in the number theory to the task being solved in interferometry, it can be interpreted as follows. If linear periodic functions are set by their entire values and their periods are relative primes, then a linear periodic function with the period equal to the product of the periods of these functions can also be put in single-valued correspondence with them, and this function determines the total phase difference.

Let the total phase difference

$$\phi = 2\pi N + \varphi \tag{20}$$

where N is an entire interfringe and φ is a phase shift within 2π.

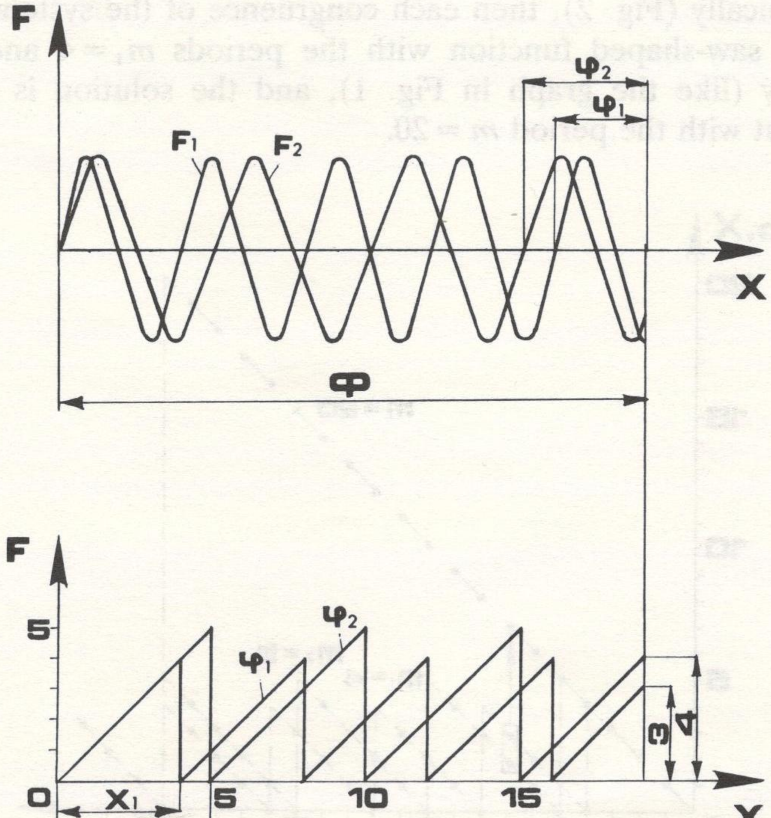


Fig. 3. Total phase difference measurement for a period ratio 5/4.

One can determine the total phase difference ϕ, not knowing N and measuring only φ, due to the method proposed.

Let the value of ϕ be measured as shown in Fig. 3. The measurement is carried out with two different periods of interference fringes X_1 and X_2 ($X_2/X_1 = 5/4$). The phases of two harmonic functions F_1 and F_2 in parts corresponding to the periods are also shown in Fig. 3. They have a kind of linear periodic function. It is necessary to regard these functions as the integer saw-shaped ones shown in Fig. 2. Measuring the phase within one period only, one has $\varphi_1 = 3$ and $\varphi_2 = 4$, respectively (Fig. 3). Then it follows from Fig. 2 that the total phase difference $\phi = 19$ in the same units, i.e. 4·75 of period X_1 or 3·75 of period X_2.

As another example, two computer-generated patterns of phase fields are shown in Fig. 4(a) and (b). They are obtained for the ratio of

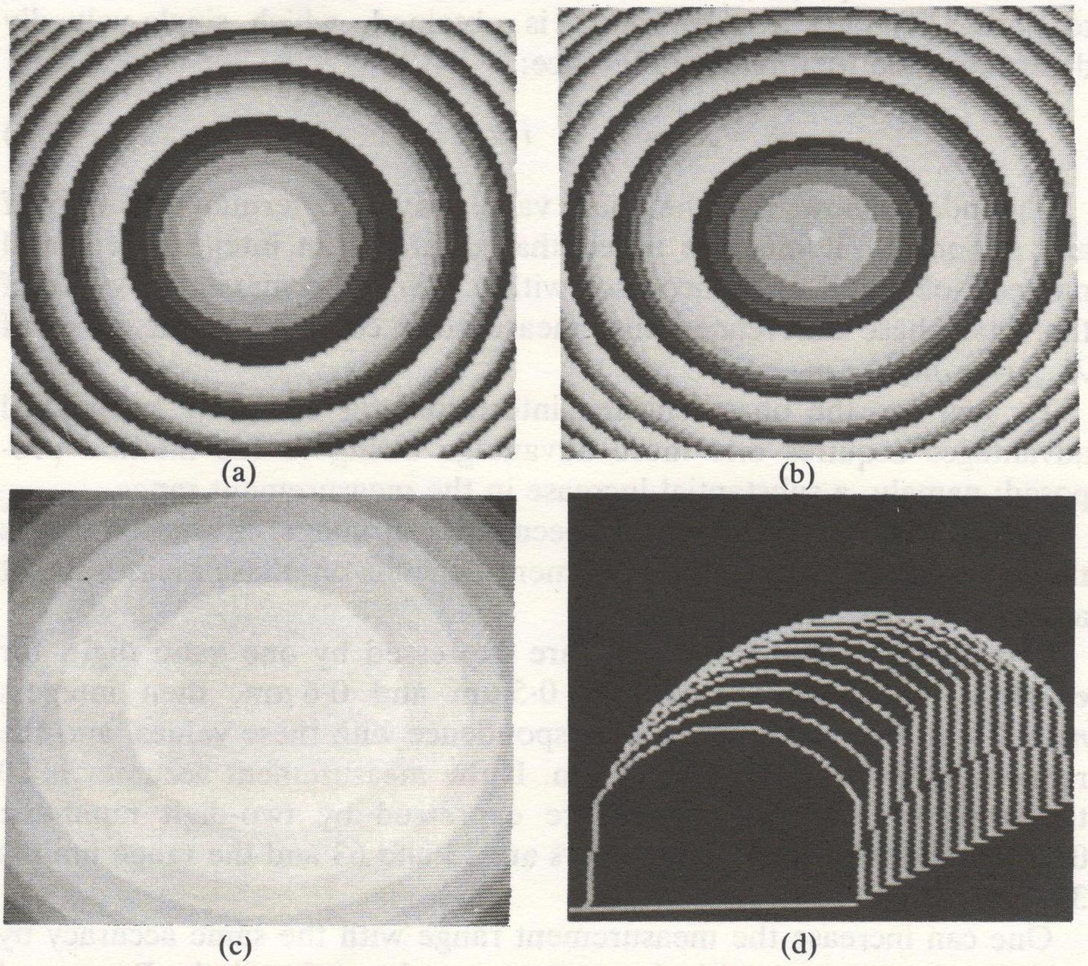


Fig. 4. (a) and (b) Two computer generated patterns for an interfringe period ratio 473/502. (c) Phase map. (d) Three dimensional contour map.

interfringe periods 473/502. Such a ratio will be further used in the physical experiment. For this case a solution like eqn (12) is the following congruence:

$$X \equiv 131022b_1 + 106425b_2 \pmod{237446} \tag{21}$$

The congruence, eqn (21), is used to unwrap the phase ambiguities and, as a result, a phase map is presented in Fig. 4(c). Figure 4(d) shows the same phase map in the form of a three-dimensional diagram.

Thus, if measurements are carried out by the phase shifting method with different periods of interfringes $X_1, X_2, \ldots, X_k$, then one obtains phase shifts $\varphi_1, \varphi_2, \ldots, \varphi_k$, respectively. Integers $m_1, m_2, \ldots, m_k$ and $b_1, b_2, \ldots, b_k$ with a register length determined by the precision of the measurements are put in correspondence with these values. The integers $m_1, m_2, \ldots, m_k$ must be relative primes. Now a system of congruences, eqn (13) whose right parts equal $b_1, b_2, \ldots, b_k$ and whose moduli equal $m_1, m_2, \ldots, m_k$, respectively, is solved. According to eqns (14)–(17) a value of X is obtained, which single-valuedly determines the total phase difference:

$$\phi_i = X/m_i \qquad i = 1, 2, \ldots, k \tag{22}$$

The index i shows that the phase value may be determined in units of any period m_i. It must be noted that ϕ_i is not an integer but a real number, obtained to an accuracy within $1/m_i$. The maximum value of the total phase difference being measured is confined to the value of $X_{max} = m_1 m_2 \cdots m_k$.

To sum up, the phase shifting interferometer alongside recognized advantages acquires one more advantage owing to the method proposed; namely, a substantial increase in the measurement range.

A very interesting and specific peculiarity of integer interferometry is the fact that the range of measurement depends on phase measurement accuracy within 2π.

If the periods and the phases are expressed by one valid digit, for example, interfringe spaces are $0{\cdot}5\,\mu$m and $0{\cdot}6\,\mu$m, then integers $m_1 = 5$ and $m_2 = 6$ are put in correspondence with these values, and the range limit is $m_1 m_2 = 30$, i.e. $3\,\mu$m. If the measurement accuracy is 10 times higher and the periods are expressed by two-digit numbers: $0{\cdot}53\,\mu$m and $0{\cdot}63\,\mu$m, then integers are 53 and 63 and the range limit is 3339, i.e. $33{\cdot}39\,\mu$m.

One can increase the measurement range with the same accuracy by taking measurements with three or more values of periods. By way of example, if one adds to the two periods $0{\cdot}53\,\mu$m and $0{\cdot}63\,\mu$m, as in the previous case, a third period $0{\cdot}43\ \mu m$, then the range increases to nearly $1500\ \mu m$.

It should be noted that, on the face of it, the technique of absolute measurement of block gauges using Koster's interferometer is like the proposed method. However, the two methods are essentially different. It is supposed *a priori* that a length of block gauge is known with error $\Delta l \leq \lambda_1\lambda_2/(\lambda_2 - \lambda_1)$ and one measures the value Δl with Koster's interferometer. Two phases with periods λ_1 and λ_2 may be put in mutually single-valued correspondence with Δl, and a length of block gauge can be obtained by adding $l = l_0 + \Delta l$.

For comparison of the integer interferometer and Koster's interferometer one considers the same situation, when two periods are used: $\lambda_1 = 0{\cdot}529\ \mu m$ and $\lambda_2 = 0{\cdot}633\ \mu m$.

The measurement range of Koster's interferometer is $\Delta l_{max} = 0{\cdot}529 \times 0{\cdot}633/(0{\cdot}633 - 0{\cdot}529) \approx 3{\cdot}2\ \mu m$ and the range of the integer interferometer is $529 \times 0{\cdot}633 \approx 335\ \mu m$.

EXPERIMENTAL RESULTS

Two interferograms are shown in Fig. 5(a) and (b). They were obtained using a Twyman–Green interferometer for two interfringe periods, the ratio of the first period to the second one is 4/3. The interfringe spaces are changed by turning the mirror in the reference interferometer arm. For the ratio 4/3 the accuracy of the phase determination will suffice as 1/4 in the first case and 1/3 in the second case. Such accuracy can be obtained by eye, without a phase shift method.

By using eqns (12)–(17) one has $m_1 = 4$ and $m_2 = 3$ and solves the congruence system

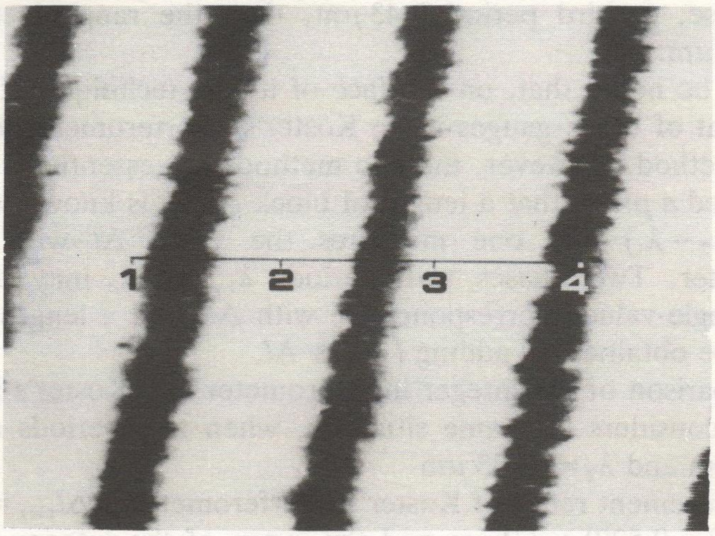
$$X \equiv b_i \pmod 4$$
$$X \equiv b_2 \pmod 3$$

The result is as follows:

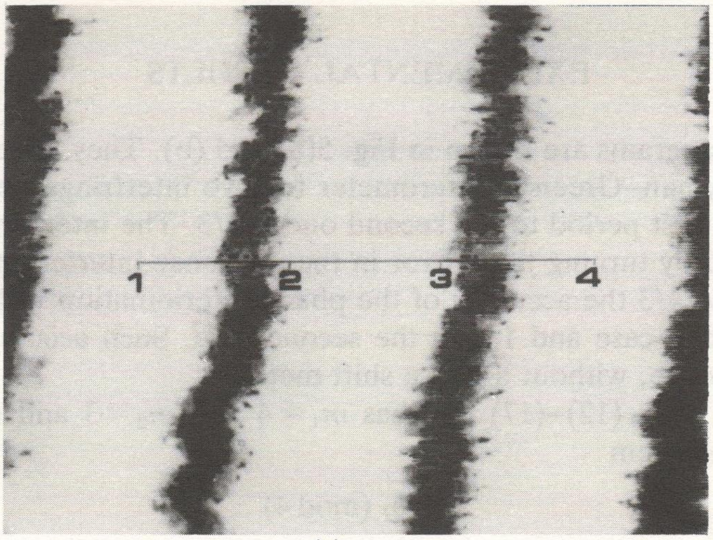
$$X \equiv 9b_1 + 4b_2 \pmod{12} \tag{23}$$

The interferograms in Fig. 5 show the points where the phase differences of two interwaves are determined. For these points the integer phase values are found within 2π. Evidently, from Fig. 5 the pair of values b_1 and b_2 are as follows:

$$b_1 = 1; 0; 3; 2$$
$$b_2 = 0; 2; 1; 0$$



(a)



(b)

Fig. 5. Two interferograms for an interfringe period ratio 4/3.

On substituting the values of b_1 and b_2 into eqn (23) one determines, respectively:

$$X = 9; 8; 7; 6$$

These integer values are the total phases at the given points when the object under study is tilted. Expressing the total phases by way of the quantity of interfringes one obtains for a space equal to four:

$$\phi = 2{\cdot}25; 2; 1{\cdot}75; 1{\cdot}5$$

respectively.

It should be noted that one has an absolute value of phase difference, not knowing a zero fringe and not counting interfringes. In this example the measurement limit is three interfringes in Fig. 5(a) or four interfringes in Fig. 5(b).

The following experiment also was performed with the help of a Twyman–Green interferometer but using a controllable phase shift and different wavelengths. For generation of various wavelengths an argon laser Spectra-Physics 2020 was used. In this experiment two wavelengths were taken: 472·7 nm and 501·7 nm. The phase shift is changed by the mirror attached to the PZT. For every wavelength four interferograms were obtained. The series of fringe patterns for 472·7 nm is shown in Fig. 6. The interferograms were digitized and 256 × 256 × 8 bit sequences were recorded for subsequent processing.

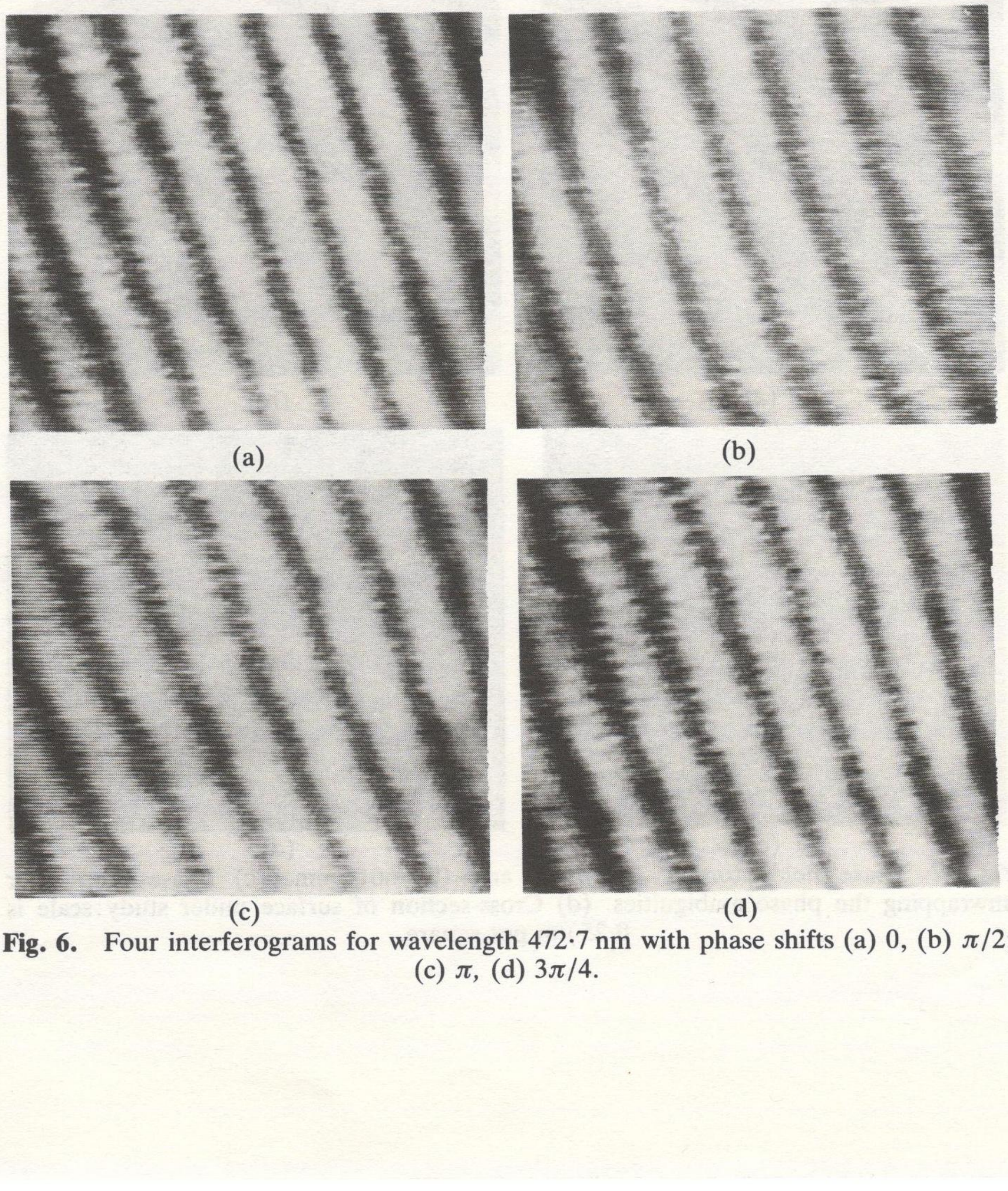


Fig. 6. Four interferograms for wavelength 472·7 nm with phase shifts (a) 0, (b) $\pi/2$; (c) π, (d) $3\pi/4$.

The phase maps were determined using the algorithm in eqns (5) and (6). Here the same problems arise as when using a two-wavelength phase shifting interferometer,[12] although the methods are essentially different. The largest errors can be attributed to chromatic aberration. It should not move the fringes or change the size of the interferogram by more than a pixel in order for good results to be obtained. The results are presented in Fig. 7(a) and (b) for 472·7 nm and 501·7 nm wavelengths, respectively.

Now moduli $m_1 = 473$ and $m_2 = 502$ are put in correspondence with these wavelengths, and eqn (21) is used for unwrapping the phase ambiguities. In this case a theoretical measurement limit is 273·5 μm. In

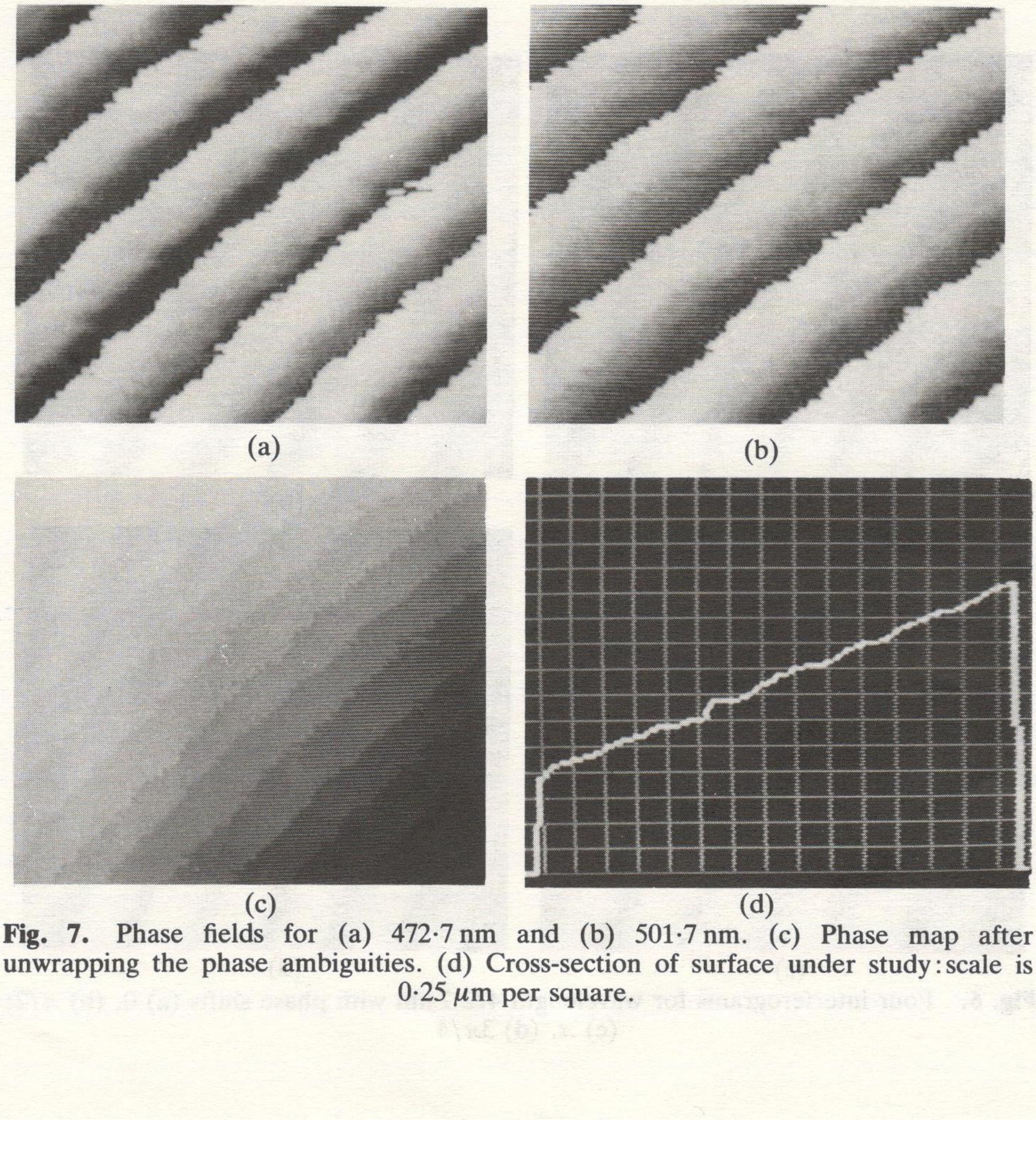


Fig. 7. Phase fields for (a) 472·7 nm and (b) 501·7 nm. (c) Phase map after unwrapping the phase ambiguities. (d) Cross-section of surface under study: scale is 0·25 μm per square.

practice, the limit was about 20 μm, and we could use the method which we called 'correction of flagrant errors' for obtaining the exact values.[10] Figure 7(c) shows the field of the total phase difference and Fig. 7(d) presents the cross-section of surface under study.

CONCLUSION

The data presented in this paper demonstrate the successful use of a new method—integer interferometry—for substantial extension of the measurement limits of the well-known method—phase shifting interferometry. The solutions of a congruence system are used for unwrapping the phase ambiguities. Computer simulations and physical experiments confirm the efficiency of the method proposed. A significant advantage of integer interferometry is a complete automatic processing of fringe patterns with any shape and degree of complexity.

REFERENCES

1. Hariharan, P., Oreb, W. F. & Brown, N., A digital phase-measurement system for real-time holographic interferometry. *Optics Communication,* **41** (6) (1982) 393–7.
2. Carre, P., Installation et utilisation du comparateur photoelectrique et interferential du Bureau Internetional des Poids et Mesures. In *Metrologia,* **2** (10) (1966), 13–23.
3. Hariharan, P., Oreb, B. F. & Eiju, T., Digital phase-shifting interferometry: a simple error-compensating phase calculation algorithm. *Applied Optics,* **26** (13) (1987) 2504–6.
4. Bruning, J. H., Herriet, D. R., Gallagher, J. E., Rosenfeld, D. P., White, A. D. & Brangaccio, D. J., Digital wavefront measurement interferometer for testing optical surfaces and lenses. *Applied Optics,* **13** (11) (1974) 2693–703.
5. Kinnstactter, K., Lohmann, A. W., Schwider, J. & Streibl, N., Accuracy of phase shifting interferometry. *Applied Optics,* **23** (24) (1988) 5082–9.
6. Nakadate, S. & Saito, H., Fringe scanning speckle-pattern interferometry. *Applied Optics,* **24** (14) (1985) 2172–80.
7. Vinogradov, I. M., *Bases of the Number Theory*. Nauka, 1972.
8. Gregory, R. T. & Krishnamurthy, E. V., *Methods and Applications of Error-Free Computation.* Springer-Verlag, 1984.
9. Gushov, V. I. & Solodkin, Yu. N. Using integer properties for interferogram analysis. *Optica & Spectroscopia* **65** (5) (1988) 1123–8.

10. Gushov, V. I. & Solodkin, Yu. N. Estimate of integer interferometer precision. *Optica & Spectroscopia* **65** (6) (1988) 1313–6.
11. Gushov, V. I. & Solodkin, U. N., Using integer properties for automatic processing of fringe patterns. In *Physical Research Fringe '89 Automatic Processing of Fringe Patterns,* ed. W. Osten, R. J. Pryputniewicz, G. T. Reid & H. Rottenkolber. Akademie-Verlag, Berlin, 1989, pp. 76–8.
12. Creath, K., Cheng, Y.-Y. & Wyant, J. C., Contouring aspheric surfaces using two-wavelength phase-shifting interferometry. *Optica Acta,* **32** (12) (1988) 1455–64.